Matrix

A matrix is a rectangular array of numbers.

The <u>size</u> of a matrix is written $\underset{\#\text{ of rows}}{m} \times \underset{\#\text{ of columns}}{n}$.

Ex: $A = \begin{bmatrix} 3 & 4 & 0 & 1 \\ -1 & 2 & 3 & 4 \\ 0 & 3 & 9 & 1 \end{bmatrix}$.

A has 3 rows $A = [\equiv]$

and 4 columns $A = [\,|\,|\,|\,|\,]$.

So A is a 3×4 matrix.

These are the rows of A: $[3\ 4\ 0\ 1]$, $[-1\ 2\ 3\ 4]$, $[0\ 3\ 9\ 1]$

These are the columns of A: $\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$.

Observe The rows of A are vectors in $\mathbb{R}^4$

the columns of A are vectors in $\mathbb{R}^3$.

In general the rows of an $m \times n$ matrix are vectors in $\mathbb{R}^n$ and

the columns are vectors in $\mathbb{R}^m$.

Echelon Form of a Matrix

ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.[2]
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix}$.

Ex a replacement

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix} \quad R3 \rightarrow (-1)R1 + R3 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

Ex an interchange

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 5 \\ 0 & 1 & 2 \end{bmatrix} \quad R2 \leftrightarrow R3 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

Ex a scaling

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix} \quad R3 \rightarrow \tfrac{1}{5} R3 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

DEFINITION

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

A matrix in <u>echelon form</u> looks like:

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

where $\blacksquare$ = non-zero number
$*$ = any number

A matrix in <u>reduced row echelon form</u> or <u>rref</u> looks like:

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

<u>Note</u> rref of a matrix is unique

The left-most non-zero entry in a each row of a matrix in echelon or rref is called a pivot position.

The number of leading 1's / pivot positions is called the rank of A.

Computing Ax

Let A be a $m \times n$ matrix. Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be vector in $\mathbb{R}^n$.

Two ways to compute $A\vec{x}$.

Method 1 - the row method

$$\underset{m \times n}{\overset{A}{\begin{bmatrix} - \text{row } 1 - \\ - \text{row } 2 - \\ \vdots \\ - \text{row } m - \end{bmatrix}}} \underset{n \times 1}{\overset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}} = \underset{m \times 1}{\overset{A\vec{x}}{\begin{bmatrix} \text{row } 1 \cdot \vec{x} \\ \text{row } 2 \cdot \vec{x} \\ \vdots \\ \text{row } m \cdot \vec{x} \end{bmatrix}}}$$

where $\text{row } i \cdot \vec{x}$ is the dot product

Method 2 - the column method

$$\underset{m \times n}{\overset{A}{\begin{bmatrix} | & | & & | \\ \text{col } 1 & \text{col } 2 & \cdots & \text{col } n \\ | & | & & | \end{bmatrix}}} \underset{n \times 1}{\overset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}} = \underset{m \times 1}{\overset{A\vec{x}}{x_1 \begin{bmatrix} | \\ \text{col } 1 \\ | \end{bmatrix} + x_2 \begin{bmatrix} | \\ \text{col } 2 \\ | \end{bmatrix} + \dots + x_n \begin{bmatrix} | \\ \text{col } n \\ | \end{bmatrix}}}$$

Ex Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

Method I: $A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 2 \cdot (-1) + 3 \cdot 2 \\ 4 \cdot 3 + 5 \cdot (-1) + 6 \cdot 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 19 \end{bmatrix}$

Method II: $A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 12 \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \end{bmatrix} + \begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 7 \\ 19 \end{bmatrix}$

Method I and II give the same result. ✓

Method I and II give the same result. ✓

Method I shows that if $\vec{x}$ satisfies $A\vec{x} = \vec{0}$, then $\vec{x}$ is orthogonal to the rows of A.

Method II shows that $A\vec{x}$ is a linear combination of columns of A.

Let A be $m \times n$ matrix. For $\vec{x} \in \mathbb{R}^n$ we get that

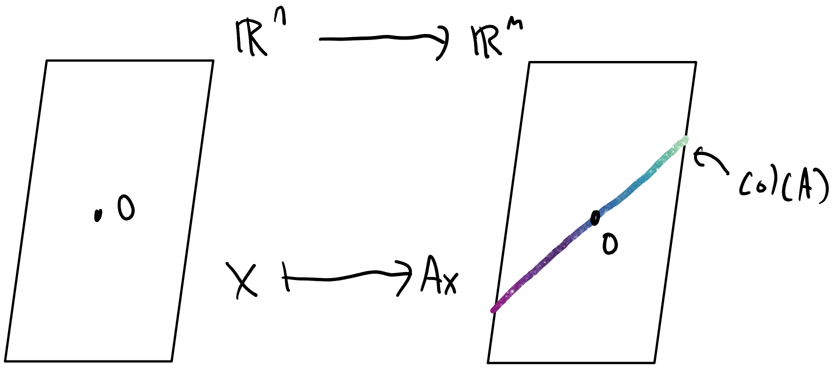
$$\underset{m\times n}{A}\ \underset{n\times 1}{\vec{x}} = \underset{m \times 1}{A\vec{x}}$$

So the action of multiplying A by $x \in \mathbb{R}^n$ results in a vector $Ax \in \mathbb{R}^m$.

So A is a function from $\underbrace{\mathbb{R}^n}_{\text{domain}} \longrightarrow \underbrace{\mathbb{R}^m}_{\text{co-domain}}$

The range of A is the set of all linear combinations of the columns of A = col(A).

$$\{ A\vec{x} \mid \text{for all } \vec{x} \in \mathbb{R}^n \} = \text{Span of columns} = \text{col}(A).$$



Question: What is the dimension of col(A)? When will col(A) equal all of R^m?

Answer: The size of col(A) is equal to rank(A) or the number of pivots in an echelon form of A. col(A) equals R^m when rank(A)=m.

Let A be a $m \times n$ matrix. Then the following are subspaces determined by A.

Column Space The column space of a matrix is the span of its columns.

Write $A = [\underbrace{v_1 \cdots v_n}_{\text{colums of } A}]$. Then $\text{col}(A) = \text{span}\{v_1, \dots, v_n\}$
$= \{c_1 v_1 + \cdots + c_n v_n \mid c_i \in \mathbb{R}\}$

$\dim \text{col}(A) = \text{rank}(A)$

A basis of col(A) is found by identifying the pivot positions of an echelon form of A and selecting the corresponding columns from the original A.

Row Space The row space of a matrix is the span of its rows.

Write $A = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}$. Then $\text{row}(A) = \text{span}\{r_1, \dots, r_m\}$
$= \{c_1 r_1 + \cdots + c_m r_m \mid c_i \in \mathbb{R}\}$

$\dim \text{row}(A) = \text{rank}(A)$

A basis for row(A) is computed by selecting the non-zero rows of an echelon form of A.

Null Space The null space of a matrix is the solution set to $Ax = \vec{0}$.

$\dim \text{nul}(A) = n - \text{rank}(A)$

A basis for nul(A) is found by decomposing the parametrized general solution x for $Ax = \vec{0}$.

Example The matrix $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -2 & 3 \\ -1 & 0 & -3 & -2 \end{bmatrix}$ is row equivalent to $B = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Find bases for col(A), row(A), and nul(A).

Sol'n Since B is in rref and row equivalent to A, B = rref(A).

rref(A) has pivots in columns 1, 2, so a basis for col(A) is columns 1, 2 from A.

rref(A) has pivots in columns 1, 2, so a basis for col(A) is columns 1, 2 from A.

$\rightarrow$ basis for col(A) = $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

$\rightarrow$ basis for row(A) = $\{ \underbrace{[1\ 0\ 3\ 2],\ [0\ 1\ -2\ 3]}_{\text{the non-zero rows of rref(A)}} \}$

$\rightarrow$ basis for nul(A)

Solve $Ax = \vec{0}$. $[A\ \vec{0}] = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ -1 & 0 & -3 & -2 & 0 \end{array}\right] \end{array}$

$\Big\downarrow$ rref

$[B\ \vec{0}] = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$

x_1, x_2 basic variables

x_3, x_4 free variables (no pivots)

Row 1: $x_1 = -3x_3 + 2x_4$

Row 2: $x_2 = 2x_3 - 3x_4$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 + 2x_4 \\ 2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \quad x_3, x_4 \in \mathbb{R}$$

basis for nul(A)

Consider A, a $m \times n$ matrix.

$$A = \begin{array}{c c} & n \text{ columns} \\ m \text{ rows} & \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \end{array} \begin{array}{c} \vec{x} \\ \begin{bmatrix} \\ \\ \\ \end{bmatrix} \end{array} \leftarrow n \times 1$$

rows of A are in $\mathbb{R}^n$

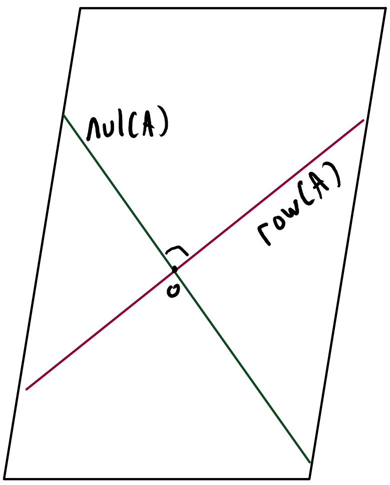
columns of A are in $\mathbb{R}^m$.

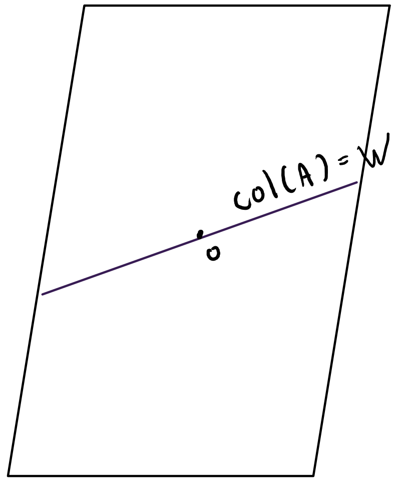
Observe If $x \in \mathbb{R}^n$ then $Ax \in \mathbb{R}^m$.

So A gives a mapping:

$$\mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\underset{\in \mathbb{R}^n}{x} \longmapsto \underset{\in \mathbb{R}^m}{Ax}$$





Subspaces of $\mathbb{R}^n$:

row A = span of rows of A

nul A = solutions to $A\vec{x} = \vec{0}$

Every vector in row(A) is orthogonal to every vector in nul(A) and vice versa.

Subspaces of $\mathbb{R}^m$:

col A = span of columns of A

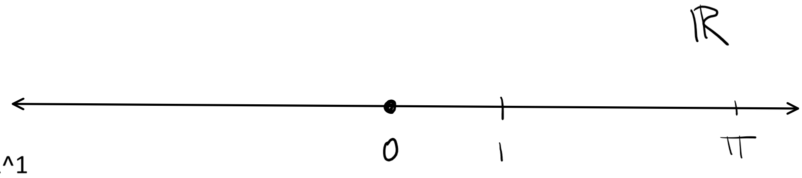
Dimensions:

$\dim(\text{row} A) = \text{rank}(A)$

$\dim(\text{nul} A) = n - \text{rank}(A)$

$\dim(\text{col} A) = \text{rank}(A)$

Subspaces of R^n



Subspaces of R^1

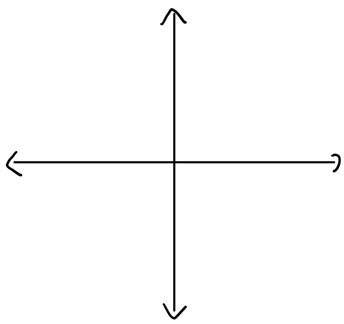
Dim=0 subspaces of R^1

- $\{0\}$ zero

Dim=1 subspaces of R^1

- all of $\mathbb{R}$
- basis is any non zero #

Subspaces of R^2



Dim=0 subspaces of R^2

- $\{\vec{0}\}$ zero

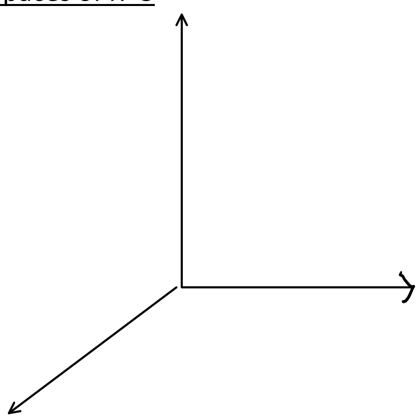
Dim=1 subspaces of R^2

- any line through the origin
- basis is any non zero vector in $\mathbb{R}^2$

Dim=2 subspaces of R^2

- $\mathbb{R}^2$
- basis is any 2 independent vectors in $\mathbb{R}^2$

Subspaces of R^3



Dim=0 subspaces of R^3

- $\{\vec{0}\}$ zero

Dim=1 subspaces of R^3

- any line through the origin
- basis is any non zero vector in $\mathbb{R}^3$

Dim=2 subspaces of R^3

- any plane through the origin
- basis is any 2 independent vectors in $\mathbb{R}^3$

Dim=3 subspaces of R^3

- $\mathbb{R}^3$
- basis is any 3 independent vectors in $\mathbb{R}^3$

Subspaces of R^n

Dim=0 subspaces of R^n

- $\{\vec{0}\}$ zero

Dim = q where q = 1, 2, … , n-1 subspaces of R^n

- any q-hyperplane through the origin
- basis is any q independent vectors in $\mathbb{R}^n$

Dim=n subspaces of R^n

- $\mathbb{R}^n$
- basis is any n independent vectors in $\mathbb{R}^n$

<u>Note that:</u>

(i) The span of any set of vectors is a subspace.

(ii) Any subspace is the span of some set of vectors (it has a basis).

So this is a comprehensive list all subspaces.

Dimension of subspace Symmetric 2x2 matrices

Let $V = M_{2,2}$ = vector space of 2×2 matrices.

What is the dimension of W, the subspace of <u>symmetric</u> 2×2 matrices?

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. A is symmetric if $A = A^T$.

So for $A \in W$ we need $A = A^T$ or $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$$\Downarrow$$

$a = a$ $\quad$ $b = c$

$d = d$ $\quad$ $c = b$

So $A \in W$ looks like $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$

$$W = \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ for } a,b,d \in \mathbb{R} \right\}$$

3 basis vectors $\rightarrow$ $\dim(W) = 3$

Let $v_1 = (1,2,3)$ $v_2 = (4,5,6)$. And let $A = [v_1\ v_2] = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.
Let $\vec{b} = (-1,1,3)$.

The following 3 questions are equivalent:

Question 1 Is $\vec{b}$ a linear combination of v_1 and v_2?

Question 2 Is $\vec{b} \in \text{span}\{v_1, v_2\} = \text{col}(A)$?

Question 3 Does the matrix equation $A\vec{x} = \vec{b}$ have a solution?

Answer Q3: $A\vec{x} = \vec{b}$ looks like:

$$\underset{A}{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}}$$

Solve for unknown $\vec{x}$ by forming the augmented matrix and row reduce

$$[A \vdots b] = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & \end{array} \\ \left[\begin{array}{cc:c} 1 & 4 & -1 \\ 2 & 5 & 1 \\ 3 & 6 & 3 \end{array}\right] \end{array} \xrightarrow{\text{rref}} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & \end{array} \\ \left[\begin{array}{cc:c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

Solution is
$x_1 = 3$
$x_2 = -1$.

Answer to Question 1,2,3:

Yes, $A\vec{x} = \vec{b}$ has a solution: $\vec{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ works.

Yes, $\vec{b}$ is a linear combination of v_1, v_2 since

$$\overset{x_1}{\overset{\shortparallel}{3}} v_1 + \overset{x_2}{\overset{\shortparallel}{(-1)}} v_2 = \vec{b}$$

which also shows $\vec{b} \in \text{span}\{v_1, v_2\}$. ✓

Question 4

Using the same v_1, v_2 as above, is $\vec{c} = (1,1,3)$ in span$\{v_1, v_2\}$?

Solve augmented matrix

$$[A \vdots \vec{c}] = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc:c} 1 & 4 & 1 \\ 2 & 5 & 1 \\ 3 & 6 & 3 \end{array}\right] \end{array} \xrightarrow{\text{rref}} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \end{array}$$

$\downarrow$

NO SOL'N!

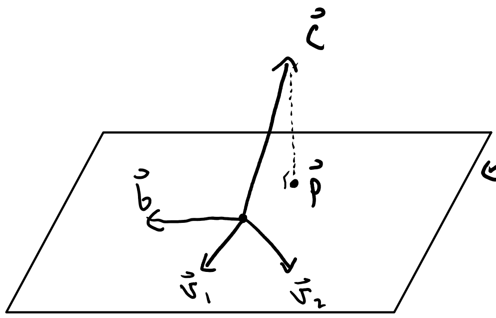
If you ever see a row like $[\underbrace{0\ 0\ \cdots\ 0}_{\text{zero's}}\ \underset{\text{non-zero}}{*}]$

while reducing an augmented matrix, then the system has no solution.

Answer to Question 4 No, $\vec{c}$ is not in span$\{v_1, v_2\}$.

We showed this by showing $A\vec{x} = \vec{c}$ has no solution.

Picture



span$\{v_1, v_2\}$ = W = col(A) fills a 2-dimensional plane through origin

$\vec{b}$ is in the plane, $\vec{c}$ is outside the plane

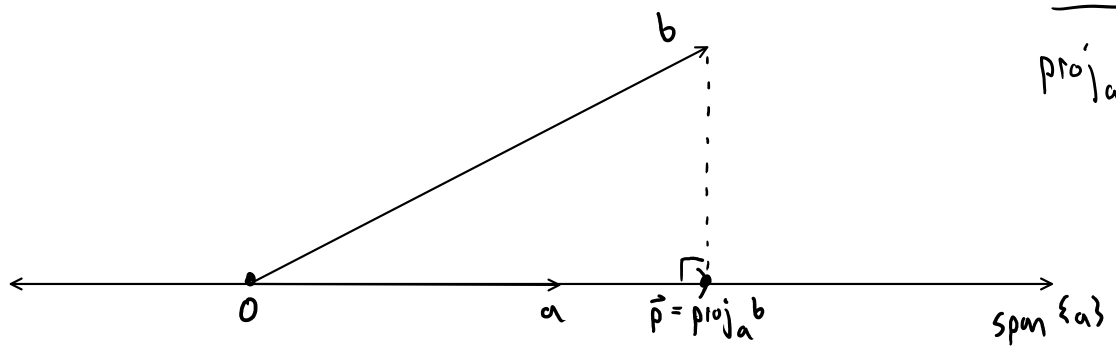
Question

Since c is not in W=col(A), we can then ask: What is the vector in W=col(A) closest to c? That is the projection of c onto W.

Before projecting onto a 2-dimensional subspace we need to project onto a 1-dimesional line.

Projection onto a line

Consider $a, b \in \mathbb{R}^n$. The projection of b onto a is the vector in span$\{a\}$ closest to b. Denote this vector $\vec{p} = \text{proj}_a b$.



Formula:

$$\text{proj}_a b = \frac{b \cdot a}{a \cdot a} \vec{a}$$

Projection onto a plane

Before projection onto a plane, we need to find an orthogonal basis for the plane W and then use the theorem:

THEOREM 6.3.4 *Let W be a finite-dimensional subspace of an inner product space V.*
(*a*) *If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthogonal basis for W, and $\mathbf{u}$ is any vector in V, then*

$$\text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r \qquad (12)$$

Our given v1 and v2 are NOT orthogonal since their dot product is not zero. To create an orthogonal basis that spans W, use the Gram-Schmidt process.

Exercise: Use the Gram-Schmidt process and compute the projection of c onto W.

Gram-Schmidt procedure

transforms an independent set of vectors into an orthogonal set that spans the same space.

The Gram–Schmidt Process

Given a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}$$

Then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthogonal basis for W. In addition

$$\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\} \qquad \text{for } 1 \le k \le p \tag{1}$$

$$\begin{tikzcd}
\text{Basis for } W \; \{x_1, \dots, x_p\} \arrow[r, bend left, "\text{Gram-Schmidt}"] & \text{Orthogonal basis for } W \; \{v_1, \dots, v_p\} \arrow[r, bend left, "\text{Normalize}"] & \text{Orthonormal basis for } W \; \{u_1, \dots, u_p\} \\
\text{Dependent spanning set } \{y_1, \dots, y_n\} \arrow[u, "\text{rref to indentify indep. vectors}"'] & &
\end{tikzcd}$$